TMA4195 Mathematical Modelling — Autumn 2018

Student Project - Group 6

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Abstract

This project explores the dynamics of a glacier and describes its shape and internal flow. Conservation of mass and momentum is applied in order to obtain a system of equations that describes the dynamics of the glacier. The equations are then rescaled, and a number of approximations are applied in order to simplify the model. Expressions for the flow field are obtained in a similar way, and applied in order compute the flow field numerically. The finite volume method was successfully implemented and used to model advancing and retreating glaciers.

Introduction

Glaciers are moving objects even though they might look like solid blocks of ice. In reality they melt, flow, grow and collapse, holding immense powers that has carved out fjords and shaped mountains for thousands of years. In this project we will model the dynamics of a glacier, looking closer at the shape, size and internal movements.

We will assume that our glacier is situated along a valley with constant and relatively small incline α, where x^* denotes the position along the length of the valley, and z^* denotes the height perpendicular to the valley floor. An assumption is also made on the behavior being uniform along the width of the valley, effectively making y^* irrelevant in the model.

The z^*-coordinate of the glacier's surface at time t^* and position x^* is given by $h^* = h^*(x^*, t^*)$. The point velocity of the glacier is denoted by $\boldsymbol{w}^* = (u^*, v^*)$, with u^* and v^* being the x^* and z^* velocity components, respectively.

A system of equations describing the relationship between the height of the glacier, its internal dynamics and accumulation rate is derived using the principles of conservation of mass and momentum. The system of equations is reduced to single partial differential equation by the use of a series of approximations. Equations for the velocity are also obtained.

Method

Ice Sheet Modelling

The Eulerian formulation of the law of conservation of mass is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_0}\rho(x,t)\,\mathrm{d}x\bigg|_{t=t_0}+\int_{\partial R_0}\rho(x,t_0)(\boldsymbol{w}^*\cdot\boldsymbol{n})\,\mathrm{d}\sigma=\int_{R_0}q^*(x,t_0)\,\mathrm{d}x, \tag{1}$$

where R_0 is a given domain and ∂R_0 its boundary. The outer unit normal vector of the domain boundary surface is given by $\boldsymbol{n}$, while q^* is production of mass over the domain and ρ denotes the density. The differential form of this equation is

$$\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\boldsymbol{w}^*)=q^*. \tag{2}$$

Assuming constant density ρ results in $\frac{\partial\rho}{\partial t}=0$ and $\nabla\cdot(\rho\boldsymbol{w}^*)=\rho\nabla\cdot\boldsymbol{w}^*$. By also assuming no internal production, $q^*=0$, one arrives at

$$\nabla\cdot\boldsymbol{w}^*=0. \tag{3}$$

Conservation of momentum and Newton's second law of motion implies that the velocity of the glacier must satisfy

$$\begin{cases}\rho\partial_{t^*}u^*+\rho\nabla u^*\cdot\boldsymbol{w}^*=-\partial_{x^*}p^*+\nabla\cdot\tau^*_x+f_x,\\ \rho\partial_{t^*}v^*+\rho\nabla v^*\cdot\boldsymbol{w}^*=-\partial_{z^*}p^*+\nabla\cdot\tau^*_z+f_z.\end{cases} \tag{4}$$

The body forces on each point are denoted by $\boldsymbol{f}=(f_x,f_y)$, p^* denotes the pressure, and

$$\tau^*=\begin{bmatrix}\tau^*_x\\ \tau^*_z\end{bmatrix}=\begin{bmatrix}\tau^*_{xx} & \tau^*_{xz}\\ \tau^*_{zx} & \tau^*_{zz}\end{bmatrix} \tag{5}$$

is the stress tensor in two dimensions. The pressure p^* can be written as a sum;

$$p^*(x^*,z^*,t^*)=\rho g\cos(\alpha)\cdot(h^*(x^*,t^*)-z^*)+\tilde{p}^*(x^*,z^*,t^*), \tag{6}$$

with g being the gravitational acceleration at the surface of the earth. The first term in (6) is the hydrostatic pressure within the glacier, and the second term is the counterpressure induced by the constant density ρ.

We will now model the body forces within the glacier. First observe that the two first terms of the left hand side of equation (4) are the time and space derivatives of momentum. We can make a simplification by assuming that gravity and friction changes are much greater than momentum and pressure changes. These two terms can therefore be approximated to 0. The pointwise gravitational force $\boldsymbol{f_g}$ is given by

$$\boldsymbol{f}_g=\begin{bmatrix}f_{gx}\\ f_{gz}\end{bmatrix}=\begin{bmatrix}\rho g\sin(\alpha)\\ \rho g\cos(\alpha)\end{bmatrix}. \tag{7}$$

Under the assumption that the dominating body force is f_g, equation (4) is simplified to:

$$\begin{cases} -\partial_{x^*} p^* + \nabla \cdot \tau_x^* + \rho g \sin(\alpha) = 0, \\ -\partial_{z^*} p^* + \nabla \cdot \tau_z^* + \rho g \cos(\alpha) = 0. \end{cases} \tag{8}$$

The space derivatives of the pressure sum given in equation (6) can be inserted into equation (8), yielding

$$\begin{aligned} \nabla \cdot \tau_x^* + \rho g \sin(\alpha) - \rho g \cos(\alpha) \cdot \partial_{x^*} h^* - \partial_{x^*} \tilde{p}^* &= 0, \\ \partial_{x^*} \tau_{xz}^* - \partial_{z^*} \tau_{xx}^* - \partial_{z^*} \tilde{p}^* &= 0. \end{aligned} \tag{9}$$

We will now develop a model for the stress tensor τ^*, with the (common) assumption that the stress tensor is related to the strain rate by *Glen's law*:

$$\begin{aligned} \partial_{x^*} u^* &= \mu(\theta^*)^{m-1} \tau_{xx}^*, \\ \partial_{z^*} v^* &= \mu(\theta^*)^{m-1} \tau_{zz}^*, \\ \frac{1}{2}(\partial_{z^*} u^* + \partial_{x^*} v^*) &= \mu(\theta^*)^{m-1} \tau_{xz}^* = \mu(\theta^*)^{m-1} \tau_{zx}^*. \end{aligned} \tag{10}$$

Here,

$$\theta^* := \left(\frac{1}{2} {\tau_{xx}^*}^2 + {\tau_{xz}^*}^2 + \frac{1}{2} {\tau_{zz}^*}^2 \right)^{1/2}, \tag{11}$$

where μ and m are material constants depending on, amongst others, the temperature of the ice. Common values for the exponent is $m \in [1.8\,,\; 5]$, with $m = 3$ being a typical choice. Conservation of angular momentum and the assumption of incompressibility gives

$$\mathrm{Tr}(\tau) = \tau_{xx} + \tau_{zz} = 0 \tag{12a}$$

$$\tau_{xz} = \tau_{zx}, \tag{12b}$$

which inserted into equation (11) yields:

$$\theta^* := \left({\tau_{xx}^*}^2 + {\tau_{xz}^*}^2 \right)^{1/2}. \tag{13}$$

If we find the divergence of the stress tensor and apply these assumptions, insertion into (9) yields

$$\begin{aligned} \partial_{x^*} \tau_{xx}^* + \partial_{z^*} \tau_{xz}^* + \rho g \sin(\alpha) - \rho g \cos(\alpha) \cdot \partial_{x^*} h^* - \partial_{x^*} \tilde{p}^* &= 0, \\ \partial_{x^*} \tau_{xz}^* - \partial_{z^*} \tau_{xx}^* - \partial_{z^*} \tilde{p}^* &= 0. \end{aligned} \tag{14}$$

After all these simplifications, we end up with the remaining unknowns u^*, v^*, h^*, $\tilde{p}^*$, τ_{xx}^*, and θ^*. At the surface, $z^* = h^*(x^*, t^*)$, the force from the atmospheric pressure is still accounted for by $\tilde{p}^*$. We assume that the sheer stress between the glacier and the atmosphere is negligible. Then $\mathrm{d}\boldsymbol{F} = \boldsymbol{T} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}\sigma = 0 \Rightarrow \tau_{xx}\hat{n}_x + \tau_{xz}\hat{n}_z = 0$, and $\tau_{zz}\hat{n}_x + \tau_{zx}\hat{n}_z = 0$. By (12a) and (12b), $(\tau_{xx} - \tau_{xx})\hat{n}_x + 2\tau_{xz}\hat{n}_z = 0$, giving $\tau_{xz} = 0$. At the bottom, $z^* = 0$, we simplify the problem by assuming that the glacier is frozen along the entirety of the valley floor, resulting in $\boldsymbol{w}^*|_{z^*=0} = 0$.

In order to find an expression for the glacier height $h^*(x^*, t^*)$, let the accumulation of ice, in length pr. unit time, be denoted by $q^*(x^*,t^*)$. The density is given as mass pr. unit area and is constant throughout the glacier. Let R be a control area extending over the interval $[x^*, x^*+\epsilon]$, from the bottom of the glacier to the surface. Let $\boldsymbol{n}$ be the outer unit normal vector of this control area. By the construction of the control volume the term $\boldsymbol{w}^* \cdot \boldsymbol{n}$ reduces to $\pm u^*$. Applying conservation of mass (1) within R with this in mind, the equation can be rewritten as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{x^*}^{x^*+\epsilon} h^*(x^*,t^*)\ \mathrm{d}x \\ &+\left[\int_0^{h^*(x^*+\epsilon,t^*)} u^*(x^*+\epsilon,z^*,t^*)\ \mathrm{d}z^* - \int_0^{h^*(x^*,t^*)} u^*(x^*,z^*,t^*)\ \mathrm{d}z^*\right] \\ &= \int_{x^*}^{x*+\epsilon} q^*(x^*,t^*)\ \mathrm{d}x. \end{aligned} \tag{15}$$

Note that the second term in (15) can be written as $f(x^*+\epsilon)-f(x^*)$, where $f(x^*) = \int_0^{h^*(x^*,t^*)} u^*(x^*,z^*,t^*)\ \mathrm{d}z^*$. Dividing by ϵ and letting $\epsilon \to 0$ gives us the definition of the derivative. Thus let us rewrite the term as $\frac{\mathrm{d}}{\mathrm{d}x^*}\int_0^{h^*(x^*,t^*)} u^*(x^*,z^*,t^*)\ \mathrm{d}z^*$. The first term can be rewritten as $\frac{\mathrm{d}}{\mathrm{d}t}\left[H^*(x^*+\epsilon,t^*) - H^*(x^*,t^*)\right]$, where $H^*(x^*,t^*)$ is the anti-derivative of $h^*(x^*,t^*)$. Dividing by ϵ and letting $\epsilon \to 0$ produces the derivative of $H^*(x^*,t^*)$, resulting in $\frac{\mathrm{d}}{\mathrm{d}t}h^*(x^*,t^*)$. The same technique can be used for the third term.

By using the results above we get the following equation when dividing the whole equation by ϵ and letting $\epsilon \to 0$:

$$\frac{\partial h^*}{\partial t^*} + \frac{\mathrm{d}}{\mathrm{d}x^*}\int_0^{h^*(x^*,t^*)} u^*(x^*,z^*,t^*)\,\mathrm{d}z^* = q^*(x^*,t^*). \tag{16}$$

Shallow Ice Approximations

In this subsection, we will take into account the specific geometry of a glacier. We will do so by rescaling the equations in an appropriate way and identifying small terms, presuming that the length L of a glacier is typically much larger than its height H. We choose the natural scales

$$x^* = Lx \quad \text{and} \quad z^* = Hz$$

and try to make use of the fact that $\epsilon := H/L \ll 1$. We scale the height with the same scale H;

$$h^* = Hh.$$

To find suitable scales V, U and T for the velocities and the time, such that

$$u^* = Uu, \quad v^* = Vv \quad \text{and} \quad t^* = Tt,$$

we balance the terms in the equations for conservation of mass. Using these scales, and rewriting (3) suggests that $\frac{U}{L} = \frac{V}{H}$. Next we insert $q^* = Qq$ into (16) and rewrite as

$$\frac{H}{T}\frac{\partial h}{\partial t} + H\frac{U}{L}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^h u\,\mathrm{d}z = Qq.$$

To balance the equations we set $T = \frac{H}{Q}$, $U = \frac{QL}{H}$ and $V = Q$, so (16) becomes

$$\frac{\partial h}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}x}\int_0^h u\,\mathrm{d}z = q. \tag{17}$$

We now use the scales

$$\Theta = \Theta_{xz} = \rho g H\sin(\alpha) \quad \text{and} \quad \Theta_{xx} = P = \frac{H}{L}\Theta = \epsilon\Theta$$

for the stresses θ^*, τ^*_{xz} and τ^*_{xx} and the pressure $\tilde{p}^*$, balancing the other equations in our system. Rescaling the equations (3), (10), (13) and (14), we obtain

$$\partial_x u + \partial_z v = 0, \tag{18a}$$

$$\frac{\Theta}{L}\epsilon\partial_x\tau_{xx} + \frac{\Theta}{H}\partial_z\tau_{xz} + \rho g\sin(\alpha) - \rho g\cos(\alpha)\epsilon\partial_x h - \frac{\Theta}{L}\epsilon\partial_x\tilde{p} = 0, \tag{18b}$$

$$\partial_x\tau_{xz} - \partial_z\tau_{xx} - \partial_z\tilde{p} = 0, \tag{18c}$$

$$\frac{Q}{H}\partial_x u = \mu\Theta^m\epsilon\theta^{m-1}\tau_{xx}, \tag{18d}$$

$$\frac{1}{2}\left(\frac{Q}{H}\frac{1}{\epsilon}\partial_z u + \frac{Q}{L}\partial_x v\right) = \mu\Theta^m\theta^{m-1}\tau_{xz}, \tag{18e}$$

$$\theta = \left(\epsilon^2\tau_{xx}^2 + \tau_{xz}^2\right)^{1/2}. \tag{18f}$$

Performing a first order asymptotic on equation (18b) we obtain $\Theta\partial_z\tau_{xz} \approx -\rho g H\sin(\alpha) = -\Theta \Rightarrow \partial_z\tau_{xz} \approx -1$. Equation (18f) gives $\theta \approx |\tau_{xz}|$. With this we obtain from equation (18e) $\partial_z u \approx \partial_z u + \epsilon^2\partial_x v = \kappa|\tau_{xz}|^{m-1}\tau_{xz}$, where $\kappa = 2H\mu\Theta^m\epsilon/Q$. A first order expansion of equation (18d) simply gives $\partial_x u = 0$. Using that $\partial_z\tau_{xz} = -1$ and $\tau_{xz}(h) = 0$ we get that $\tau_{xz} = -z + h$. By inserting this into the approximation for $\partial_z u$, and using that $z \in [0, h]$ we get that $\partial_z u = \kappa(-z+h)^m$.

Now let $d(x)$ denote the height profile of the boundary, such that $0 < d(x) < h(x,t)$, so integrating and combining this with $u(x, d(x), t) = 0$ leaves us with

$$u = \frac{\kappa}{m+1}\left((h-d(x))^{m+1} - (h-z)^{m+1}\right). \tag{19}$$

Inserting this expression for u into equation (17) and performing the integration, leaves us with

$$\frac{\partial h}{\partial t} + \lambda\frac{\mathrm{d}}{\mathrm{d}x}\left(h-d(x)\right)^{m+2} = q, \tag{20}$$

where $\lambda = \frac{\kappa}{m+2}$. Inserting (19) into equation (18a), integrating with respect to z and using the boundary condition $v(x,d(x),t)=0$ results in

$$v=-\frac{\partial}{\partial x}\frac{\kappa}{m+1}\left[(z-d(x))(h-d(x))^{m+1}+\frac{1}{m+2}\left([h-z]^{m+2}-(h-d(x))^{m+2}\right)\right]. \tag{21}$$

Notice that if we assume a constant height profile $d(x)\equiv 0$, this will only change the limits and initial conditions when integrating over the height, and not the fundamental dynamics inside the glacier. From here the discussion will be restricted to $d(x)\equiv 0$.

In the previous asymptotic expansions of equations (18) we did not treat α as a small parameter in the same way as ϵ. If we assume $\gamma=\epsilon\cot(\alpha)\sim 1$, we still obtain a relatively simple PDE for h. With this new assumption about α, first order asymptotic expansion of equation (18a) gives $\partial_z\tau_{xz}=-\partial_x h$. Thus $\tau_{xz}=h_x(h-z)$, $u=\frac{\kappa}{m+1}h_x^m\left[h^{m+1}-(h-z)^{m+1}\right]$, leading to the equation

$$\frac{\partial h}{\partial t}+\lambda\frac{\mathrm{d}}{\mathrm{d}x}\left(h_x^m h^{m+2}\right)=q. \tag{22}$$

Dynamics Of Glaciers

A steady state height, denoted by $\tilde{h}$, is identified by setting $\frac{\partial h}{\partial t}=0$. Combining this with our expression from (20) leaves us with $\frac{\mathrm{d}}{\mathrm{d}x}\tilde{h}^{m+2}=\frac{q}{\lambda}$. Integrating and solving for $\tilde{h}$ leaves us with the following equation for identifying steady states:

$$\tilde{h}(x)=\left(h_0^{m+2}+\int_0^x\frac{1}{\lambda}q(y)\,\mathrm{d}y\right)^{\frac{1}{m+2}}. \tag{23}$$

The height of the glacier at $x=0$ is given by $\tilde{h}(x)\mid_{x=0}=h_0$ and is the only required boundary condition for the equation. The toe of a glacier in a steady state can be found by setting $\tilde{h}(x)=0$ and finding:

$$x_f=\min\left\{x:\tilde{h}(x)=0\right\}. \tag{24}$$

We will now consider a slight perturbation of a steady state. Starting with a steady state $\tilde{h}(x)$ for some accumulation rate $q=q(x)$, i.e. $\partial_t\tilde{h}=0$, we have from equation (20) that $\lambda\partial_x\tilde{h}^{m+2}=q$. We now assume a small perturbation $h_\delta=\tilde{h}+\delta k$, where $\delta>0$ is a small parameter. Inserting into equation (20), performing a Taylor expansion of h_δ^{m+2} around $\tilde{h}$ gives and ignoring $o(\delta)$ terms, gives

$$\delta\partial_t k+\delta\frac{\mathrm{d}}{\mathrm{d}x}g(\tilde{h},k)+o(\delta)=0, \tag{25}$$

where $g(\tilde{h},k)=\kappa\tilde{h}^{m+1}k$. The $o(\delta)$ term is obtained assuming k is smooth.

Ignoring terms of order $o(\delta)$ and using the method of characteristics gives the characteristic equation $x=\kappa\tilde{h}^{m+1}t+x_0$, resulting in a shock somewhere to the right of the endpoint of the glacier x_f since $\kappa\tilde{h}^{m+1}>0$ for $x<x_f$. This is not a problem, since the area to the right of x_f is not of interest. Along the characteristics, $\partial_t k=-\kappa(m+1)\tilde{h}^m k\frac{\mathrm{d}}{\mathrm{d}x}\tilde{h}$, where $-\kappa(m+1)\tilde{h}^{m+1}<0$.

Thus, at points where $\frac{\mathrm{d}}{\mathrm{d}x}\tilde{h}<0$, the steady states are unstable along the characteristics starting from x where $0<x<x_f$. However, since all the characteristics move to the right, the characteristics from the boundary $x=0$ will be the only characteristics in the area of interest after a certain point in

time. We do not consider perturbations in x_0, since this would result in a new steady state. Thus k is zero along these characteristics, and nice behaviour should be expected at least after some time. Numerically no instability is observed at all.

Numerical experiments

CFL-condition: For an explicit scheme for the differential equation $\partial h/\partial t + \lambda \partial h/\partial x = 0$ the CFL-condition is $C = \left|\lambda \frac{\Delta t}{\Delta x}\right| < 1$. However, we want to simulate the behaviour of (20), which can be written as

$$\partial_t h + \lambda(m+2)h^{m+1}\frac{\mathrm{d}}{\mathrm{d}x}h = q. \tag{26}$$

If we assume that q is sufficiently small and can be approximated as locally constant, this term can be neglected. A reasonable bound for the maximum value of the scaled height, namely $h_{\max} < 2$, can be used. The CFL-condition therefore becomes:

$$C = \left|\lambda(m+2)h_{\max}^{m+1}\frac{\Delta t}{\Delta x}\right| = \left|\kappa h_{\max}^{m+1}\frac{\Delta t}{\Delta x}\right| < \left|\kappa 2^{m+1}\frac{\Delta t}{\Delta x}\right| < 1.$$

Numerical Schemes: Applying a naive upwind scheme to (20) yields,

$$h_i^{j+1} \approx h_i^j + \Delta t q_i - \frac{\kappa \Delta t}{\Delta x}\left(h_i^j\right)^{m+1}\left(h_i^j - h_{i-1}^j\right). \tag{27}$$

This naive upwind scheme is unable to model an advancing glacier in our specific application. A finite volume method is more suitable for such a scenario. By using the finite volume method on equation (20) we obtain the scheme

$$h_i^{j+1} \approx h_i^j + \frac{\Delta t}{2}\left(q_i^j + q_{i+1}^j\right) - \frac{\lambda \Delta t}{\Delta x}\left(\left[h_i^j\right]^{m+2} - \left[h_{i-1}^j\right]^{m+2}\right). \tag{28}$$

Notice that only the last two terms change for each time step, leading to a reduction in computation time. The following simple model is used for the accumulation rate $q(x)$, with the snow line x_s and x_f positive values such that $x_s < x_f$;

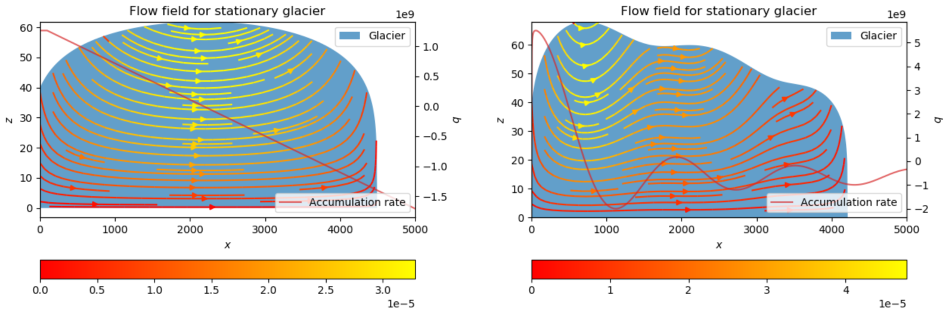
$$q(x) = \begin{cases} q_0, & x < x_s \\ q_0 + \alpha(x - x_s), & x_s < x < x_f \\ 0, & x > x_f \end{cases}. \tag{29}$$

We want to find α such that the accumulation model results in a steady state with the glacier toe situated at x_f. By inserting equation (29) into equation (23) for the steady state height, and requiring that $h(x_f) = 0$ and $h(0) = h_0$, we obtain

$$\alpha = -2\frac{q_0 x_f + (\lambda h_0)^{m+2}}{(x_f - x_s)^2}. \tag{30}$$

The explicit expressions for the flow field are shown in (19) and (21). The flow field within the glacier can be computed by evaluating these equations on grid points within the glacier. The derivative in

(21) is approximated using central differences. Forward and backwards differences are used on the boundaries of the grid. The stationary glacier for a given accumulation rate is found using the method described in (23).



(a) Accumulation rate as specified in (29) **(b)** Arbitrary accumulation rate.

Figure 1: Stationary glaciers with height on left axis, accumulation rate on right axis and stream lines for the internal flow field.

The flow field for a glacier where the accumulation rate is $q^*(x) = 50000 \sin(\frac{x}{250})\frac{1}{(x+1)} - 20$, $[q^*(x)] = \text{m year}^{-1}$, is shown in figure 1b, and the median of the velocity calculated at each of the grid points is 343 meters pr. year.

The flow field for a glacier with accumulation rate as in (29), with $q_0^* = 40$, is shown in figure 1a. The median of the velocity found at each of the grid points is 544 meters pr. year.

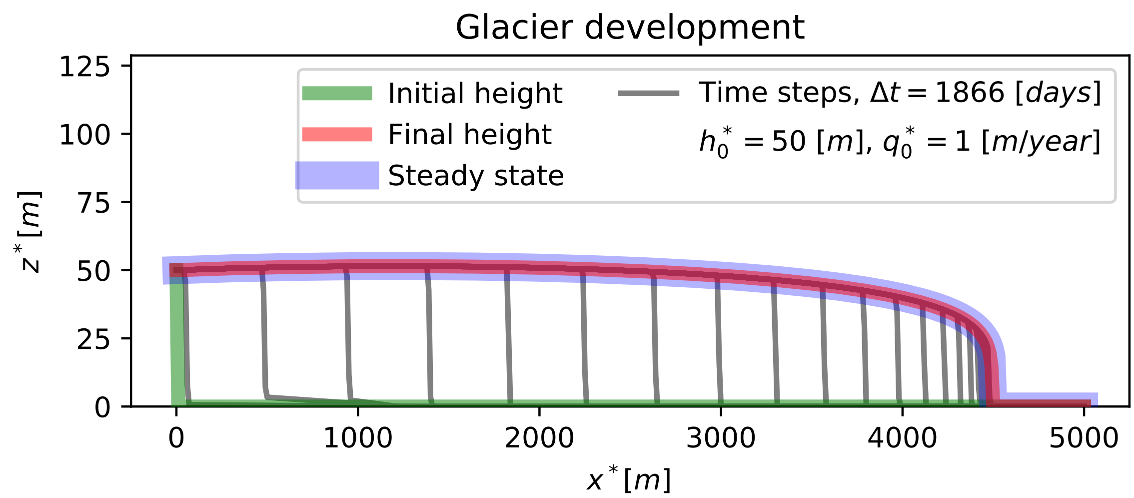


Figure 2: Advancing glacier with height on the y-axis and position along the length of the valley on the x-axis. The green line is the initial height, the blue line is the steady state, the red line is the final height of the scheme and the grey line is advancement of the glacier in equally spaced timesteps.

The visualized flow fields found for the stationary glaciers shows that matter tends to be transported deep into glacier before surfacing closer to the end of the glacier. The velocity varies depending on multiple factors, but is especially high near the top of the glacier. Less movement closer to the bottom agrees with our assumption of a glacier that is frozen at the bottom. The velocity is typically a few

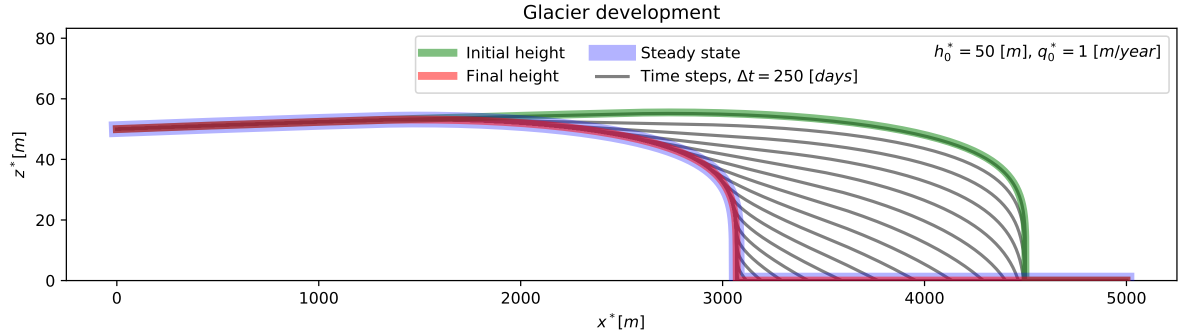


Figure 3: Retracting glacier with height on the y-axis and position along the length of the valley on the x-axis. The glacier is melting from the initial height, (green line), through the the equally spaced timesteps, (grey lines), to the final height, (red line), which is equal to the steady state, (blue line).

L	H	ρ	m	μ	q_0
1000 [m]	50 [m]	917 [kg m^{-3}]	3 [1]	9.3×10^{-21} [s^{-1} Pa^{-3}]	1 [m year^{-1}]

Table 1: Table of numerical values.

hundred meters pr. year. This value might seem high when considering that the glacier is in fact a massive block of ice, but seems more reasonable when it is taken into account that the glacier itself can move at speeds approaching a hundred meters pr. year.

Figure 2 and figure 3 show the numerical results of implementing the scheme (28) and modelling an advancing and a retreating glacier using the values in table 1. In figure 2 we see that the glacier has almost a rectangular shape while it advances. This can, at least partly, be explained by the flux in starting at a high level and remaining constant throughout the simulation. In effect a glacier which starts out with a vertical wall at $x^* = 0$ is being modelled. We also see clearly at time step two and three that there is some accumulation, but that the effect of this is being dominated by the flux while the glacier advances. It is possible that some of the simplifying assumptions made throughout this report also contribute to this behaviour.

When the steady state is obtained, the slope close to the toe is still larger than what seems natural. This may be due to the assumption of zero pressure changes, which is not a reasonable assumption when the height differences are large. Modelling pressure differences in this area, the lower parts would be pushed to the right, resulting in a smaller slope.

In figure 3 the initial height is a steady state with the snow line at $x^* = 2500$m. Then the snow line is moved to $x^* = 1250$m, and we model how the glacier behaves after such a sudden change in accumulation rate. We chose α such that the toe of the glacier in the initial state is at $x^* = 4500$m. The time step lines depicting the glacial motion from initial to final state looks more fluent for the retracting glacier than that for advancing method. Therefore our finite volume scheme seems to work better for the retracting glacier.

The code used to produce the results in this section can be found at `https://github.com/JakobGM/mathematical-modelling`.

Conclusion

We were able to model the creation and expansion of advancing glaciers. We were also able to model retreating glaciers, depicting scenario where the temperature rises and the glacier is melting at an accelerated rate.

The finite volume method, resulting in the scheme (28), was proven most successful. For areas with no accumulation, the upwind scheme, seen in (27), is unable to form an advancing glacier. If $h_i^j = 0$, the only contribution to h_i^{j+1} will come from the accumulation, which is zero, resulting in h_i^{j+1} being zero for all consecutive iterations.

Some problems occurred for small values of μ, making our model unstable. However, finding a good reference for μ was difficult, so for this project a value giving stable numerical results was chosen.

The results presented arise from a really simple accumulation model. Real life climate and temperature is time dependent, unstable and ever changing. This model neglects such complexities, and the results should therefore be taken with a large grain of salt.

The modelling of the future development of glaciers in the real world is not feasible with the techniques presented in this report. It would, for instance, require a much more sophisticated model for accumulation. The periodicity of seasons would have to be accounted for, and the boundary conditions might change as well.

The authors of this paper still want to argue that the developed model is not worthless. It is able to describe the degree of glacial retraction and/or expansion that ensues after an arbitrary accumulation change. It is mainly unable to provide *exactly* what such accumulation changes might be and what might cause it. A suggested improvement for our model is therefore the incorporation of climate forecasts, specifically with the respect to temperature and precipitation changes in glacial regions.