TMA4180 Optimization 1 — Spring 2017

Student Project

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Abstract

The theory related to a specific instance of the inverse kinematic problem is investigated and it is argued that BFGS is one of the more suitable method for solving this problem. The method is implemented with a couple of mechanisms handling specific aspects of the kinematic problem, one being the case of stationary points that are not global minima. Numerical evidence indicating superlinear convergence is presented.

Introduction

This paper is an investigation of the properties relating to an inverse kinematic problem and its solution. Specifically, we consider a robot arm that consists of n joints connected by n segments of different lengths. The last segment is connected to an end-effector. We are interested in minimizing the distance from this end-effector to a given point by finding an optimal configuration of the joints. This can be expressed as an optimization problem and solved by the use of a suitable numerical optimization method. It can be argued that the quasi-Newton method BFGS is particularly well suited.

In the following sections the theory is dealt with more closely and potential challenges are discussed. A numerical method is then presented and applied in an attempt to solve the problem. The results, including convergence analysis indicating superlinear convergence, are then presented and discussed.

Theory

The robot arm in question consists of n segments of length l_i going from joint i to joint $(i+1)$ for $i = 1, 2, \ldots, n-1$. The last segment goes from joint n to an end-effector. The angle between the ith segment and the $(i-1)$th segment is denoted by ϑ_i. The position of the first joint is chosen to be the origin, and we consider ϑ_1 to be the angle between the first segment and the x-axis.

The joint space, the set of all possible joint configurations, is denoted by $\mathscr{J}$. The joint space is equal to $\mathbb{R}^n$ since we don't impose any restrictions on the values of ϑ_i. The robot arm also has an associated configuration space, denoted by $\mathscr{C}$, which consists of all the points in $\mathbb{R}^2$ that can be reached by some configuration of the joints. Finding the position in the configuration space that corresponds to a position in the joint space can be expressed as a function $F : \mathscr{J} \to \mathscr{C}$. The function is

$$F(\vartheta) = \sum_{i=1}^{n} l_i \left[\cos(\sum_{j=1}^{i} \vartheta_i) \quad \sin(\sum_{j=1}^{i} \vartheta_i) \right]^T. \tag{1}$$

The index of the longest segment is denoted as m. The maximum reach of the end-effector is equal to the sum of all segment lengths $\sum_{i=1}^{n} l_i$. If the longest segment is longer than the sum of the remaining segments then there is a circular region around the origin that becomes unreachable. In such a case the configuration space is an annulus centered at the origin, otherwise it is a solid disk.

The set of points satisfying these restrictions can be summarized as

$$\mathscr{C} := \{ x \in \mathbb{R}^2 : l_m - \sum_{i=1, i \neq m}^{n} l_i \leq ||x|| \leq \sum_{i=1}^{n} l_i \}. \tag{2}$$

We are interested in finding a point in the configuration space $\vartheta \in \mathscr{J}$ which is mapped to the destination point $p \in \mathscr{C}$ by F. Solving this problem, $F(\vartheta) = p$, is equivalent to solving the optimization problem $\min\limits_{\vartheta \in \mathscr{C}} f(\vartheta)$, where $f(\vartheta) := \frac{1}{2}||F(\vartheta) - p||_2^2$. Minimizing this problem for destination points outside the configuration space have easily obtainable analytical solutions, and do not require the use of a numerical optimization algorithm.

The boundedness of the configuration space, as defined in 2, imposes the following bounds on the function being optimized.

$$0 \leq f(\vartheta) \leq 2||\sum_{i=1}^{n} l_i||_2^2, \quad \forall \vartheta \in \mathscr{J}. \tag{3}$$

Since $F(\vartheta)$ is a continuous function, $f(\vartheta)$ is continuous as well. The continuity and compactness of the optimization problem result in the problem admitting a solution.

The global minima are not unique since we haven't imposed restrictions on ϑ. Observe for example that if any of the angles are changed by a multiple of 2π the function value will remain unchanged. The problem is not convex since the function is periodic. In fact, the problem wouldn't be convex even if we restricted the joint space to $\mathscr{J} := [0, 2\pi)^n$ since saddle points exist. A saddle point will be located at $\vartheta = [\vartheta_1, 0, \pi]$ when $l = [3, 2, 2]$ and p is located somewhere along the first arm segment. This is true for all values of ϑ_1. The existence of saddle points could pose a challenge when implementing a algorithm for numerical optimization.

The gradient of the function can be expressed as

$$\nabla f(\vartheta) = \left[\frac{\partial f}{\partial \vartheta_1} \quad \frac{\partial f}{\partial \vartheta_2} \quad . \quad . \quad . \quad \frac{\partial f}{\partial \vartheta_k} \right]^T, \tag{4}$$

where we use that

$$\frac{\partial f}{\partial \vartheta_i} = \frac{\partial F_x}{\partial \vartheta_i}(F_x(\vartheta) - p_x) + \frac{\partial F_y}{\partial \vartheta_i}(F_y(\vartheta) - p_y). \tag{5}$$

We introduce the notation $a_k = \sum_{i=k}^{n} l_i \cos(\sum_{j=1}^{k} \vartheta_j)$ and $b_k = \sum_{i=k}^{n} l_i \sin(\sum_{j=1}^{k} \vartheta_j)$. The derivative of $f(\vartheta)$ with respect to ϑ_k then becomes

$$\frac{\partial f}{\partial \vartheta_k} = -b_k[a_1 - p_x] + a_k[b_1 - p_y]. \tag{6}$$

The identification of stationary points that are not global minima is rather straightforward. This is done by considering the distance between the end-effector and the destination point, as we will discuss in the next section.

When the robot arm has reached the destination it satisfies the constraint of connecting the origin to the destination point with line segments. With three or more joints and a destination point within the interior of the configuration space it is possible to continuously introduce small changes in the angles and still at all times satisfy this constraint. Thus there exist non-isolated minima.

Optimization method

Implementation of gradient descent, nonlinear conjugate gradient methods, Newton's method and the quasi-Newton method BFGS were considered. Gradient descent is in a sense the most intuitive method. However, in most cases it will not be especially efficient in comparison to the other methods mentioned. Conjugate gradient methods have an advantage when solving relatively big problems, since they do not involve solving linear systems or matrix-matrix multiplications. In practical applications we assume the robot arm to have a limited number of segments. Efficiently solving large systems is therefore not prioritized.

Newton's method is known to converge quadratically to a solution from a point sufficiently close to a strict local minimizer under specific assumptions [1, Theorem 3.5, p. 44]. Since the non-isolated minima discussed above are not strict, the theorem does not apply to the inverse kinematic problem. This does not exclude using Newton's method since there are problems where the theorem does not apply but the method is still considered to be efficient, especially if one only regards the number of iterations needed to converge to a solution. A drawback is that Newton's method has to calculate the Hessian matrix repeatedly. This often has a high computational cost, as in our case, leading us towards our method of choice. The BFGS method does not compute the actual Hessian, but rather an approximation which is always positive definite. The approximation may be computed without using matrix-matrix operations or other $O(n^3)$ operations. This is true for the rest of the algorithm as well. Thus all operations, including the evaluations of $f(\vartheta)$ and $\nabla f(\vartheta)$, can be performed at a cost of $O(n^2)$ arithmetic operations [1, p. 140].

The BFGS method has been implemented as outlined in [1, p. 140], with alterations taking into account situations where an analytical solution is available and situations where stationary points that are not global minima are visited during an instantiation. The initial Hessian is computed using the heuristic as suggested in [1, p. 143], scaling the identity matrix after the first step. The algorithm described in [1, p. 60] is used to perform a line search satisfying the strong Wolfe conditions. Since the gradient is bounded, which implies Lipschitz continuity, superlinear convergence is expected for the BFGS method [1, p. 158] without the aforementioned alterations. The exceptional case invoking the second alteration may lead to slower convergence.

Using a numerical solver to solve problems where the point p is located outside the configuration space $\mathscr{C}$ is not necessary. The case where the point lies outside the outer circle of the configuration space is easily handled by setting the angle of the first joint ϑ_1 equal to the angle between a straight line from origin to the point and the x-axis. In the other case, where the point lies inside the inner circle of an annulus, we set the direction of the longest segment to be in the same direction as in the first case. The rest of the segments are set to point in the opposite direction.

The existence of stationary points that are not critical points poses a problem when the numerical optimization method is applied. These points reduce the gradient to zero, which is often implemented as a stopping criteria. To avoid ending up in points such as these we implement a criterion which checks if the point $F(\vartheta)$ is within the tolerated distance from the destination point. However, this does not stop the numerical algorithm from moving towards stationary points. In order to escape such stationary points we use the strategy described above to detect when we're approaching a stationary point. Then a set of random vectors in the neighbourhood around the identified stationary point is generated, the vector with the smallest function value is chosen and a naive line search is applied in its direction after having asserted that the function value is smaller than in the stationary point.

Numerical experiments

Figure 1 shows a plot of the solution BFGS converged to with parameters and initial angles as described in the caption. The stopping criterion $||F(\vartheta) - p|| < 10^{-3}$ is used. Convergence to correct solutions is also observed when testing the algorithm thoroughly with a series of initial configurations and destination points. These include different kinds of stationary points in addition to random configurations.

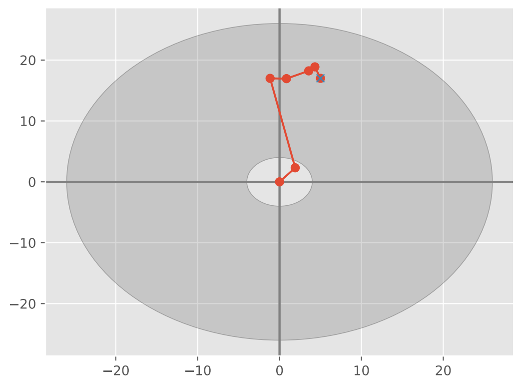
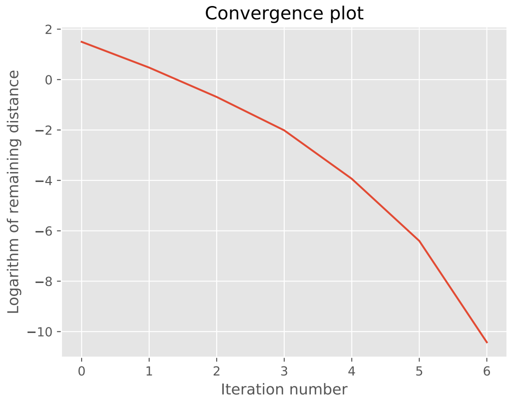
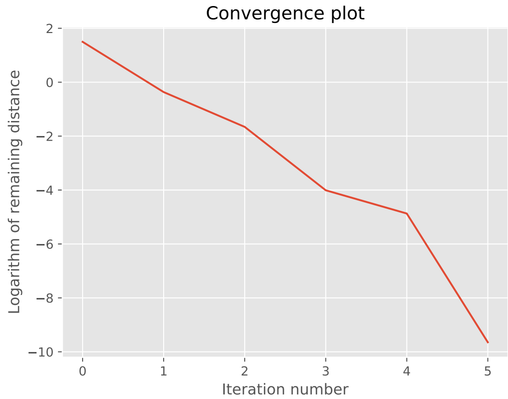


Figure 1: Convergence of the numerical method for initial $\vartheta = \mathbf{0}$, $l = (3, 15, 2, 3, 1, 2)$ and $p = (5, 17)$. The dark grey area indicates the configuration space.

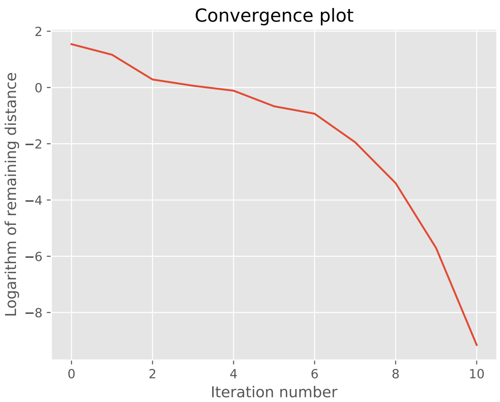
Tests of convergence are done with the problems described in the captions in figure 2. Although the method is run only for a few iterations due to rapid convergence, the plots still indicate superlinear convergence in general.



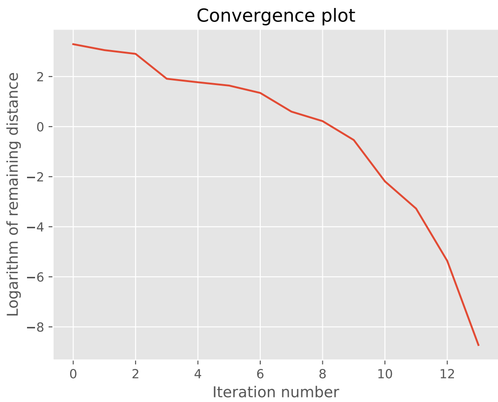
(a) $n = 3$, $l = (3, 2, 2)$, $p = (3, 2)$



(b) $n = 4$, $l = (3, 2, 1, 1)$, $p = (3, 2)$



(c) $n = 4$, $l = (3, 2, 1, 1)$, $p = (0, 0)$



(d) $n = 6$, $l = (3, 8, 2, 6, 4, 2)$, $p = (5, 18)$

Figure 2: Convergence plots for different segment lengths and desired points.

Conclusion

The BFGS method, with appropriate modifications, converges to a correct solution for all classes of problems within a reasonable time frame. Satisfactory behaviour is also observed for problems of substantially larger size than the ones presented here.

The implementation does not discriminate between different solutions to the same problem. It is possible to implement a cost function which penalizes large changes in ϑ in order to get a solution which requires less movement. One could also implement a separate cost for each separate joint, for example one could consider the cost of moving the outermost joints as less demanding than moving the innermost joint.

References

[1] J. Nocedal, *Numerical Optimization*, Springer, 2006.